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HIGH GAIN FEEDBACK SYSTEMS AS SINGULAR SINGULAR-PERTURBATION PR--ETC(U)
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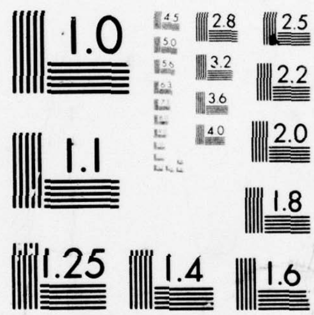

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HIGH GAIN FEEDBACK SYSTEMS AS SINGULAR SINGULAR-PERTURBATION PROBLEMS*

R. E. O'MALLEY, JR.

Department of Mathematics
University of Arizona
Tucson, Arizona 85721

Abstract

As an example of a singular singular-perturbation problem, we solve the high gain system $\dot{x} = A(t)x + B(t)u$, $u = gC(t)x$ as $g = 1/\mu \to \infty$. Our primary assumption requires BC to have fixed rank $k < n$ and no unstable eigenvalues. We find an asymptotic solution of the form $x(t,\mu) = X(t,\mu) + \Pi(\tau,\mu)$ where the fast transient $\Pi \to 0$ as $\tau = t/\mu \to \infty$.

1. Introduction

Singular perturbation methods have now become an increasingly accepted and valued new approach for solving limiting problems and obtaining simplified models in control (cf. Kokotovic et al. (1976)). Recently, Jameson and this author [cf. O'Malley and Jameson (1975, 1977)] have gained new insight on singular problems of optimal control by analyzing them as the limit of nearly singular ones. This involved the study of singularly perturbed two-point boundary value problems. Unlike the more traditional problems (as encountered with linear regulator problems with singularly perturbed state equations), the limiting behavior is not determined by the solution of a natural reduced problem obtained by setting $\varepsilon = 0$. Instead, the reduced problem has an infinite number of solutions and some other means is needed to specify which of these is the limiting solution. Motivated by this problem, and the critical need to solve analogous problems for stiff differential equations, O'Malley and Flaherty (1977) have begun to examine such "singular" singular-perturbation problems. They turn out to be interesting mathematically, in addition to their obvious practical importance.

In these proceedings, Young et al. (1977) discuss high gain feedback systems. We refer the reader to their paper for appropriate references and a discussion of the significance of such studies in various control situations. We shall examine these as an example of a singular singular-perturbation problem illustrating a very successful interplay between singular perturbations theory and its applications in control.

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We shall consider the linear high gain system

$$\begin{cases} \dot{x} = A(t)x + B(t)u, & t \geq 0 \\ u = gC(t)x \end{cases} \tag{1}$$

where the state vector x is n-dimensional, the control vector u is m-dimensional, and the scalar gain factor g is large. Introducing the small parameter

$$\mu = 1/g, \tag{2}$$

then, the feedback system (1) takes the form

$$\mu\dot{x} = (B(t)C(t) + \mu A(t))x, \quad t \geq 0. \tag{3}$$

If BC remains a stable matrix, classical singular perturbations theory implies the existence of a unique asymptotic solution to the initial value problem for (3) of the form

$$x(t,\mu) = \Pi(\tau,\mu) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\mu^j. \tag{4}$$

Here, the terms Π_j each tend to zero as the stretched variable

$$\tau = t/\mu \tag{5}$$

tends to infinity (cf. Hoppensteadt (1971) and O'Malley (1974)). In particular, away from $t = 0$, the limiting solution $X_o(t) = 0$ is the unique solution of the reduced problem

$$B(t)C(t)X_o(t) = 0. \tag{6}$$

Young et al. consider the singular problem where rank $(BC) = m < n$. Then, (6) has an $n - m$ dimensional linear manifold of solutions, though we expect the initial value problem to have a unique solution for each $\mu > 0$.

We shall make the assumption

(H) the $n \times n$ matrix $B(t)C(t)$ has a constant rank k, $0 \leq k < n$, for all t in $0 \leq t \leq T < \infty$; its null space is spanned by $n - k$ linearly independent eigenvectors; and its nonzero eigenvalues all have negative real parts there.

Then, we shall obtain a unique asymptotic solution to the initial value problem for (3) of the form

$$x(t,\mu) = X(t,\mu) + \Pi(\tau,\mu) \tag{7}$$

where the outer solution $X(t,\mu)$ and the boundary layer correction $\Pi(\tau,\mu)$ both have asymptotic series expansions in μ, with the boundary layer terms decaying to zero as the stretched variable $\tau = t/\mu$ tends to infinity.

2. Preliminary Linear Algebra

Under hypothesis (H), the matrix BC can be put into its row-reduced echelon form by use of an orthogonal matrix E, i.e. we'll obtain

$$E(t)B(t)C(t) = \begin{pmatrix} U(t) \\ 0 \end{pmatrix} \tag{8}$$

where $U(t)$ is a $k \times n$ matrix of rank k. Indeed, E can be readily obtained via Householder transformations (cf. Golub (1965)). Writing

$$E = \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{9}$$

where E_1 is $k \times n$, we have

$$E_2BC = 0 \tag{10}$$

while the orthogonality of E implies that

$$E_1E_2' = 0, \quad E_1E_1' = I_k, \quad \text{and} \quad E_2E_2' = I_{n-k}. \tag{11}$$

Further,

$$P = E_1'E_1 = P^2 \quad \text{and} \quad Q = E_2'E_2 = Q^2 \tag{12}$$

are complementary projections with $P + Q = I_n$, and Q projects into the null space of $(BC)'$ since

$$(BC)'Q = (BC)'E_2'E_2 = (E_2BC)'E_2 = 0$$

by (10). Since the dimensions of $R(Q)$, the range of Q, and $N((BC)')$, the null space of $(BC)'$, are both $n - k$, we have

$$\begin{cases} R(Q) = N((BC)') \\ \text{and} \\ R(P) = R(BC). \end{cases} \tag{13}$$

Finally, (9) and (10) imply that

$$EBCE' = \begin{pmatrix} E_1BCE_1' & E_1BCE_2' \\ 0 & 0 \end{pmatrix}, \tag{14}$$

so the $k \times k$ matrix

$$S = E_1BCE_1' \quad \text{is stable} \tag{15}$$

since it has the k stable eigenvalues of BC guaranteed by hypothesis (H).

3. A Solution via a Transformed Problem

If we make the $1 - 1$ transformation

$$w = Ex, \tag{16}$$

the variable w satisfies the singularly perturbed system

$$\mu\dot{w} = (EBCE' + \mu(\dot{E} + EA)E')w. \tag{17}$$

This, however, is also a singular singular-perturbation problem since the matrix $EBCE'$ has rank $k < n$.

The structure of the solution becomes obvious upon splitting w into vector components

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} E_1x \\ E_2x \end{pmatrix}. \tag{18}$$

They satisfy the "nonsingular" singularly perturbed system

$$\begin{cases} \mu\dot{w}_1 = Sw_1 + E_1BCE_2'w_2 + \mu\mathcal{E}_1w_1 + \mu\mathcal{E}_2w_2 \\ \dot{w}_2 = \mathcal{E}_3w_1 + \mathcal{E}_4w_2 \end{cases} \tag{19}$$

where we've used the decomposition (14) for $EBCE'$ and set

$$(\dot{E} + EA)E' = \begin{pmatrix} \mathcal{E}_1 & \mathcal{E}_2 \\ \mathcal{E}_3 & \mathcal{E}_4 \end{pmatrix}. \tag{20}$$

Standard singular perturbation theory shows that the initial value problem for (19) will have a unique solution of the form

$$\begin{cases} w_1(t,\mu) = W_1(t,\mu) + P_1(\tau,\mu) \\ w_2(t,\mu) = W_2(t,\mu) + \mu P_2(\tau,\mu) \end{cases} \tag{21}$$

where all terms have asymptotic expansions in μ and P_1 and P_2 tend to zero as $\tau \to \infty$. Further, the leading terms W_{10} and W_{20} of the outer expansion satisfy the reduced problem.

$$\begin{cases} 0 = SW_{10} + E_1BCE_2'W_{20} \\ \dot{W}_{20} = \mathcal{E}_3W_{10} + \mathcal{E}_4W_{20}, \; W_{20}(0) = w_2(0) = E_2(0)x(0). \end{cases} \tag{22}$$

Thus

$$W_{10} = -S^{-1}E_1BCE_2'W_{20} \tag{23}$$

while W_{20} uniquely satisfies

$$\dot{W}_{20} = (\mathcal{E}_4 - \mathcal{E}_3S^{-1}BCE_2')W_{20}, \quad W_{20}(0) = E_2(0)x(0). \tag{24}$$

The leading terms of the boundary layer correction satisfy

$$\begin{cases} \dfrac{dP_{10}}{d\tau} = S(0)P_{10}, \quad P_{10}(0) = E_1(0)x(0) - W_{10}(0) \\ \dfrac{dP_{20}}{d\tau} = E_3(0)P_{10}. \end{cases}$$

Thus, the decaying solutions are given by

$$\begin{cases} P_{10}(\tau) = e^{S(0)\tau}(E_1(0)x(0) - W_{10}(0)) \\ P_{20}(\tau) = -\int_\tau^\infty E_3(0)P_{10}(s)ds \\ \quad = E_3(0)S^{-1}(0)P_{10}(\tau). \end{cases} \tag{25}$$

Higher order terms follow analogously. Finally, $x = E'\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ implies that the unique solution of the original initial value problem is represented in the additive form (7) with the outer solution

$$X(t,\mu) = E_1'(t,\mu)W_1(t,\mu) + E_2'(t,\mu)W_2(t,\mu) \tag{26}$$

and the boundary layer correction

$$\begin{aligned}\Pi(\tau,\mu) = E_1'(\epsilon\tau,\mu)P_1(\tau,\mu) \\ + \mu E_2'(\epsilon\tau,\mu)P_2(\tau,\mu)\end{aligned} \tag{27}$$

which decays to zero as $\tau \to \infty$.

4. A Direct Solution

Let us now try to directly obtain a solution of the form (7). Since the boundary layer correction Π decays to zero as $\tau \to \infty$, it's necessary that the outer solution X satisfy the system (3) for all $t > 0$. Thus the outer expansion

$$X(t,\mu) \sim \sum_{j=0}^{\infty} X_j(t)\mu^j \tag{28}$$

must satisfy

$$\mu\dot{X} = (BC + \mu A)X \tag{29}$$

as a power series in μ. This requires that

$$BCX_j = \dot{X}_{j-1} - AX_{j-1} \equiv \xi_{j-1} \tag{30}$$

for each $j \geq 0$ with $\xi_{-1} \equiv 0$. Multiplying this equation by E_1 and manipulating, we obtain

$$PX_j = -\mathcal{A}BC(QX_j) + \mathcal{A}\xi_{j-1} \tag{31}$$

for $\mathcal{A} = E_1'S^{-1}E_1$. But $P + Q = I_n$ then implies that

$$X_j = \mathcal{B}QX_j + \mathcal{A}\xi_{j-1} \tag{32}$$

where $\mathcal{B}$ is the projection $\mathcal{B} = I - \mathcal{A}BC$. Thus each X_j has been successively determined up to its projection QX_j onto $N((BC)')$.

Since $QBC = 0$, consistency of (30) requires that $Q\xi_{j-1} = 0$. Since $\xi_{-1} = 0$, this Fredholm alternative argument requires that $(QX_j)^{\cdot} = (\dot{Q} + QA)X_j$ for each $j \geq 0$. Using (32), then, we finally obtain the linear differential equation

$$(QX_j)^{\cdot} = (\dot{Q} + QA)\mathcal{B}(QX_j) + (\dot{Q} + QA)\mathcal{A}\xi_{j-1} \tag{33}$$

for QX_j. We therefore find the outer expansion by solving algebraic equations (31) for PX_j and the differential equation (33) for its complement QX_j. Thus, we're able to formally obtain $X(t,\mu)$ termwise up to specifying the initial values $Q(0)X_j(0)$. Since the initial value $x(0)$ won't generally satisfy (31) at $t = 0$, we'll need the boundary layer correction Π to account for the resulting initial jump (i.e. a nonuniform convergence at $t = 0$ occurring in $R(BC)$).

By linearity, $\Pi(\tau,\mu)$ must satisfy

$$\frac{d\Pi}{d\tau} = [B(\epsilon\tau)C(\epsilon\tau) + \mu A(\epsilon\tau)]\Pi \tag{34}$$

as a power series

$$\Pi(\tau,\mu) \sim \sum_{j=0}^{\infty} \Pi_j(\tau)\mu^j \tag{35}$$

in μ. Thus, we must successively obtain decaying solutions of

$$\frac{d\Pi_j}{d\tau} = B(0)C(0)\Pi_j + \zeta_{j-1}, \quad j \geq 0, \tag{36}$$

where $\zeta_{-1} \equiv 0$ and ζ_{j-1} is generally a linear combination of preceding terms Π_ℓ with polynomial coefficients in τ. Since $QBC = 0$, it follows that $\frac{d}{d\tau}(Q(0)\Pi_j(\tau)) = Q(0)\zeta_{j-1}(\tau)$, so

$$Q(0)\Pi_j(\tau) = -Q(0)\int_\tau^\infty \zeta_{j-1}(s)ds, \quad j \geq 0 \tag{37}$$

It remains to find the complementary $P(0)\Pi_j(\tau)$'s. Multiplying (36) by $E_1(0)$, we have

$$\begin{aligned}\frac{d}{d\tau}(E_1(0)\Pi_j) &= S(0)(E_1(0)\Pi_j) \\ &+ E_1(0)(B(0)C(0)Q(0)\Pi_j(\tau) + \zeta_{j-1}).\end{aligned}$$

Integrating, then, and using (37), we obtain

$$\begin{aligned}\Pi_j(\tau) = E_1'(0)e^{S(0)\tau}E_1(0)\Pi_j(0) \\ - Q(0)\int_\tau^\infty \zeta_{j-1}(s)ds \\ + E_1'(0)\int_0^\tau e^{S(0)(\tau-r)}E_1(0)\cdot\end{aligned}$$

$$\cdot(-B(0)C(0)Q(0)\int_r^\infty \zeta_{j-1}(s)ds$$

$$+\ \zeta_{j-1}(r))dr. \tag{38}$$

Since $\zeta_{-1} \equiv 0$, $\Pi_o(\tau)$ is determined up to its initial value $E_1(0)\Pi_o(0)$, and it is exponentially decaying. Using induction, it readily follows that succeeding terms $\Pi_j(\tau)$ are exponentially decaying whatever initial values are later selected.

To completely specify the expansions generated, we must obtain the initial values $Q(0)X_j(0)$ and $E_1(0)\Pi_j(0)$ termwise. Since (37) implies that $Q(0)\Pi_j(0) = 0$, the representation (7) implies that $Q(0)X_o(0) = Q(0)x(0)$ and the limiting outer solution X_o will be completely determined as the solution of the $n - k$ th order dynamical system

$$\begin{aligned}(QX_o)^{\cdot} &= (\dot{Q} + QA)B(QX_o),\\ Q(0)X_o(0) &= Q(0)x(0)\end{aligned} \tag{39}$$

together with

$$X_o = B(QX_o). \tag{40}$$

Then, $E_1(0)\Pi_o(0) = E_1(0)x(0) - E_1(0)X_o(0)$ and (37) implies that

$$\Pi_o(\tau) = E_1'(0)e^{S(0)\tau}\,E_1(0)(x(0) - X_o(0)). \tag{41}$$

Continuing for each $j > 0$, we set

$$\begin{aligned}Q(0)X_j(0) &= -Q(0)\Pi_j(0)\\ &= Q(0)\int_0^\infty \zeta_{j-1}(s)ds\end{aligned} \tag{42}$$

and this allows us to uniquely obtain X_j termwise. Then

$$E_1(0)\Pi_j(0) = -E_1(0)X_j(0) \tag{43}$$

becomes known and we completely find the boundary layer correction term Π_j. The solution generated will, of course, agree with that obtained via the preceding transformed problem.

5. Summary

We find that the asymptotic solution of the high gain feedback problem (1), i.e., the initial value problem for (3) is of the form

$$x(t,\mu) = X_o(t) + \Pi_o(t/\mu) + O(\mu) \tag{44}$$

under hypothesis (H). Specifically, we note that the fast transient Π_o decays to zero, remaining in $R(BC)$, while the outer limit X_o satisfies $BCX_o = 0$. Moreover, X_o is determined by a dynamic system (39) in an $n - k$ dimensional space where $k = \text{rank}(BC)$. The results continue to hold for all $t \geq 0$ provided (H) remains valid and $X_o(t)$ decays exponentially to zero as $t \to \infty$. Finally, observe that (44) displays the two time scale nature of the limiting solution on $0 \leq t \leq T$.

Our results should be further related to earlier work in the literature. We note, in particular, Young et al.'s result that $CX_o = 0$ when $k = m = \text{rank}(CB)$. It implies that the control would stay bounded away from $t = 0$.

Acknowledgment: Thanks should be given to Petar Kokotovic and his co-authors for emphasizing the singular perturbation nature of large gain systems. It fit well with the theory of singular singular-perturbation problems, currently under study by this author.

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As an example of a singular perturbation problem, we solve the high gain system $\dot{x} = A(t)x + B(t)u$, $u = gC(t)x$ as $g = 1/\mu \to \infty$. Our primary assumption requires BC to have fixed rank $k < n$ and no unstable eigenvalues. We find an asymptotic solution of the form $x(t,\mu) = X(t,\mu) + \Pi(\tau,\mu)$ where the fast transient $\Pi \to 0$ as $\tau = t/\mu \to \infty$.

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